

Certicom ECC Challenge

Abstract

Certicom is pleased to present the Certicom Elliptic Curve Cryptosystem (ECC) Challenge. The first of its kind, the ECC Challenge has been developed to increase the industry's understanding and appreciation for the difficulty of the elliptic curve discrete logarithm problem, and to encourage and stimulate further research in the security analysis of elliptic curve cryptosystems.

It is our hope that the knowledge and experience gained from this Challenge will help confirm comparisons of the security levels of systems such as ECC, RSA and DSA that have been based primarily on theoretical considerations. We also hope it will provide additional information to users of elliptic curve public-key cryptosystems in terms of selecting suitable key lengths for a desired level of security.

The Certicom ECC Challenge Defined

The Challenge is to compute the ECC private keys from the given list of ECC public keys and associated system parameters. This is the type of problem facing an adversary who wishes to completely defeat an elliptic curve cryptosystem.

There are two Challenge Levels: Level I, comprising 109-bit and 131-bit challenges; and Level II, comprising 163-bit, 191-bit, 239-bit and 359-bit challenges. The 109-bit challenges are considered feasible and could be solved within a few months, while the 131-bit challenges will require significantly more resources to solve. All Level II challenges are believed to be computationally infeasible.

The Certicom ECC Challenge is preceded by some Exercises: 79-bit, 89-bit and 97-bit, respectively. These Exercises are feasible to complete given the current state of knowledge in algorithmic number theory and the computational resources available to the industry. Certicom believes that it is feasible that the 79-bit exercises could be solved in a matter of hours, the 89-bit exercises could be solved in a matter of days, and the 97-bit exercises in a matter of weeks using a network of 3000 computers.

Participants can attempt solving the Exercise and Challenge sets using one or both of two finite fields. The first involves elliptic curves over the finite field F_{2^m} (the field having 2^m elements in it), and the second involves elliptic curves over the finite field F_p (the field of integers modulo an odd prime p).

The following sections present further background on the Certicom ECC Challenge, a mathematical overview of the elliptic curve discrete logarithm problem, a detailed technical description of the Challenge, the Challenge lists and corresponding prizes, and details on how to report solutions.

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1 Introduction

1.1 Background

Since the invention of public-key cryptography in 1976 by Whitfield Diffie and Martin Hellman, numerous public-key cryptographic systems have been proposed. All of these systems rely on the difficulty of a mathematical problem for their security.

Over the years, many of the proposed public-key cryptographic systems have been broken, and many others have been demonstrated to be impractical. Today, only three types of systems should be considered both secure and efficient. Examples of such systems, classified according to the mathematical problem on which they are based, are:

1. **Integer factorization problem (IFP):** RSA and Rabin-Williams.
2. **Discrete logarithm problem (DLP):** the U.S. government's Digital Signature Algorithm (DSA), the Diffie-Hellman and MQV key agreement schemes, the ElGamal encryption and signature schemes, and the Schnorr and Nyberg-Rueppel signature schemes.
3. **Elliptic curve discrete logarithm problem (ECDLP):** the elliptic curve analogue of the DSA (ECDSA), and the elliptic curve analogues of the Diffie-Hellman and MQV key agreement schemes, the ElGamal encryption and signature schemes, and the Schnorr and Nyberg-Rueppel signature schemes.

None of these problems have been *proven* to be intractable (i.e., difficult to solve in an efficient manner). Rather, they are *believed* to be intractable because years of intensive study by leading mathematicians and computer scientists around the world has failed to yield efficient algorithms for solving them. As more effort is expended over time in studying and understanding these problems, our confidence in the security of the corresponding cryptographic systems will continue to grow.

1.2 Elliptic curve cryptosystems

Elliptic curve cryptosystems (ECC) were proposed independently in 1985 by Victor Miller [Miller] and Neal Koblitz [Koblitz]. At the time, both Miller and Koblitz regarded the concept of ECC as mathematically elegant, however felt that its implementation would be impractical. Since 1985, ECC has received intense scrutiny from cryptographers, mathematicians, and computer scientists around the world. On the one hand, the fact that no significant weaknesses have been found has led to high confidence in the security of ECC. On the other hand, great strides have been made in improving the efficiency of the system, to the extent that today ECC is not just practical, but it is the most efficient public-key system known.

The primary reason for the attractiveness of ECC over systems such as RSA and DSA is that the best algorithm known for solving the underlying mathematical problem (namely, the ECDLP) takes *fully exponential* time. In contrast, *subexponential-time* algorithms are known for underlying mathematical problems on which RSA and DSA are based, namely the integer factorization (IFP) and the discrete logarithm (DLP) problems. This means that the algorithms for solving the ECDLP become

infeasible much more rapidly as the problem size increases than those algorithms for the IFP and DLP. For this reason, ECC offers security equivalent to RSA and DSA while using far smaller key sizes.

The attractiveness of ECC will increase relative to other public-key cryptosystems as computing power improvements force a general increase in the key size. The benefits of this higher-strength-per-bit include:

- higher speeds,
- lower power consumption,
- bandwidth savings,
- storage efficiencies, and
- smaller certificates.

These advantages are particularly beneficial in applications where bandwidth, processing capacity, power availability, or storage are constrained. Such applications include:

- chip cards,
- electronic commerce,
- web servers,
- cellular telephones, and
- pagers.

1.3 Why have a challenge?

The objectives of this ECC challenge are the following:

1. To increase the cryptographic community's understanding and appreciation of the difficulty of the ECDLP.
2. To confirm comparisons of the security levels of systems such as ECC, RSA and DSA that have been made based primarily on theoretical considerations.
3. To provide information on how users of elliptic curve public-key cryptosystems should select suitable key lengths for a desired level of security.
4. To determine whether there is any significant difference in the difficulty of the ECDLP for elliptic curves over F_{2^m} and the ECDLP for elliptic curves over F_p .
5. To determine whether there is any significant difference in the difficulty of the ECDLP for random elliptic curves over F_{2^m} and the ECDLP for Koblitz curves.
6. To encourage and stimulate research in computational and algorithmic number theory and, in particular, the study of the ECDLP.

2 The Elliptic Curve Discrete Logarithm Problem (ECDLP)

This section provides a brief overview of the state-of-the-art in algorithms known for solving the elliptic curve discrete logarithm problem. For more information, the reader is referred to Chapter 3 of the *Handbook of Applied Cryptography* [MVV].

2.1 The discrete logarithm problem

Roughly speaking, the *discrete logarithm problem* is the problem of “inverting” the process of exponentiation. The problem can be posed in a variety of algebraic settings. The most commonly studied versions of this problem are:

1. *The discrete logarithm problem in a finite field (DLP)*: Given a finite field F_q and elements $g, h \in F_q$, find an integer l such that $g^l = h$ in F_q , provided that such an integer exists.
2. *The elliptic curve discrete logarithm problem (ECDLP)*: Given an elliptic curve E defined over a finite field F_q , and two points $P, Q \in E(F_q)$, find an integer l such that $lP = Q$ in E , provided that such an integer exists.

On the surface, these two problems look quite different. In the first problem, “multiplicative” notation is used: g^l refers to the process of *multiplying* g by itself l times. In the second problem, “additive” notation is used: lP refers to the process of *adding* P to itself l times.

If one casts these notational differences aside, then the two problems are abstractly the same. What is intriguing about the two problems, however, is that the second appears to be much more difficult than the first. The fundamental reason for this is that the algebraic objects in the DLP (*finite fields*) are equipped with two basic operations: addition and multiplication of field elements. In contrast, the algebraic objects in the ECDLP (*elliptic curves over finite fields*) are equipped with only one basic operation: addition of elliptic curve points. The additional structure present in the DLP has led to the discovery of the *index-calculus methods*, which have a *subexponential* running time. Elliptic curves do not possess this additional structure, and for this reason no one has been able to apply the index-calculus methods to the ECDLP (except in very special and well-understood cases). This absence of subexponential-time algorithms for the ECDLP, together with efficient implementation of the elliptic curve arithmetic, is precisely the reason that elliptic curve cryptosystems have proven so attractive for practical use.

2.2 Algorithms known for the ECDLP

This section briefly overviews the algorithms known for the ECDLP. All of these algorithms take *fully exponential time*.

The following notations are used:

- q is the order of the underlying finite field.
- F_q is the underlying finite field of order q .
- E is an elliptic curve defined over F_q .
- $E(F_q)$ is the set of points on E both of whose coordinates are in F_q , together with the point at infinity.
- P is a point in $E(F_q)$.
- n is the large prime order of the point P .
- Q is another point in $E(F_q)$.

The ECDLP is: Given q, E, P, n and Q , find an integer l , $0 \leq l \leq n - 1$, such that $lP = Q$, provided that such an integer exists.

For the remainder of the discussion, we shall only consider instances of the ECDLP for which the integer l exists.

1. *Naive exhaustive search.*

In this method, one simply computes successive multiples of P : $P, 2P, 3P, 4P, \dots$ until Q is obtained. This method can take up to n steps in the worst case.

2. *Baby-step giant-step algorithm.*

This algorithm is a time-memory trade-off of the method of exhaustive search. It requires storage for about \sqrt{n} points, and its running time is roughly \sqrt{n} steps in the worst case.

3. *Pollard's rho algorithm.*

This algorithm, due to Pollard [Pollard], is a randomized version of the baby-step giant-step algorithm. It has roughly the same expected running time ($\sqrt{\pi n / 2}$ steps) as the baby-step giant-step algorithm, but is superior in that it requires a negligible amount of storage.

Gallant, Lambert and Vanstone [GLV], and Wiener and Zuccherato [WZ] showed how Pollard's rho algorithm can be sped up by a factor of $\sqrt{2}$. Thus the expected running time of Pollard's rho method with this speedup is $\sqrt{\pi n} / 2$ steps.

4. *Distributed version of Pollard's rho algorithm.*

Van Oorschot and Wiener [VW] showed how Pollard's rho algorithm can be parallelized so that when the algorithm is run in parallel on m processors, the expected running time of the algorithm is roughly $\sqrt{\pi n} / (2m)$ steps. That is, using m processors results in an m -fold speed-up.

This distributed version of Pollard's rho algorithm is the fastest general-purpose algorithm known for the ECDLP.

5. *Pohlig-Hellman algorithm.*

This algorithm, due to Pohlig and Hellman [PH], exploits the factorization of n , the order of the point P . The algorithm reduces the problem of recovering l to the problem of recovering l modulo each of the prime factors of n ; the desired number l can then be recovered by using the Chinese Remainder Theorem.

The implications of this algorithm are the following. To construct the most difficult instance of the ECDLP, one must select an elliptic curve whose order is divisible by a large prime n . Preferably, this order should be a prime or almost a prime (i.e. a large prime n times a small integer h). The elliptic curves in the exercises and challenges posed here are all of this type.

6. *Pollard's lambda method.*

This is another randomized algorithm due to Pollard [Pollard]. Like Pollard's rho method, the lambda method can also be parallelized with a linear speedup. The parallelized lambda method is slightly slower than the parallelized rho method [VW]. The lambda method is, however, faster in situations when the logarithm being sought is known to lie in a subinterval $[0, b]$ of $[0, n - 1]$, where $b < 0.39n$ [VW].

7. *Multiple Algorithms*

R. Silverman and Stapleton [SS] observed that if a single instance of the ECDLP (for a given elliptic curve E and a base point P) is solved using Pollard's rho method, then the work done in solving this instance can be used to speed up the solution of other instance of the ECDLP for the same curve E and base point P . More precisely, solving k instances of the ECDLP (for the same curve E and base point P) takes only \sqrt{k} as much work as it does to solve one instance of the ECDLP. This analysis, however, does not take into account storage requirements.

Concerns that successive logarithms become easier can be addressed by ensuring that the elliptic parameters are chosen so that the first instance is infeasible to solve.

8. *A special class of elliptic curves: supersingular curves.*

Menezes, Okamoto and Vanstone [MOV, Menezes] and Frey and Rück [FR] showed how, under mild assumptions, the ECDLP in an elliptic curve E defined over a finite field F_q can be reduced to the DLP in some extension field F_{q^B} for some $B \geq 1$, where the number field sieve algorithm applies. The reduction algorithm is only practical if B is small — this is not the case for most elliptic curves, as shown by Balasubramanian and Koblitz [BK]. To ensure that this reduction algorithm does not apply to a particular curve, one only needs to check that n , the order of the point P , does not divide $q^B - 1$ for all small B for which the DLP in F_{q^B} is tractable ($1 \leq B \leq 2000/(\log_2 q)$ suffices).

For the very special class of *supersingular elliptic curves*, it is known that $B \leq 6$. It follows that the reduction algorithm yields a subexponential-time algorithm for the ECDLP in supersingular curves. For this reason, supersingular curves are explicitly excluded from use in the ECDSA.

9. *Another special class of elliptic curves: anomalous curves.*

Semaev [Semaev], Smart [Smart], Satoh and Araki [SA] showed that the ECDLP for the special class of anomalous elliptic curves is easy to solve. An *anomalous elliptic curve* over F_q is an elliptic curve over F_q which has exactly q points. The attack does not extend to any other classes of elliptic curves. Consequently, by verifying that the number of points on an elliptic does not equal the number of elements in the underlying field, one can easily ensure that the Smart-Satoh-Araki attack does not apply to a particular curve.

10. *Curves defined over a small field*

Suppose that E is an elliptic curve defined over the finite field F_{2^B} . Gallant, Lambert and Vanstone [GLV], and Wiener and Zuccherato [WZ] showed how Pollard's rho algorithm for computing elliptic curve logarithms in $E(F_{2^{Bd}})$ can be further sped up by a factor of \sqrt{d} -- thus

the expected running time of Pollard's rho method for these curves is $\sqrt{\pi m/d}/2$ steps. For example, if E is a Koblitz curve, then Pollard's rho algorithm for computing elliptic curve logarithms in $E(\mathbb{F}_{2^m})$ can be sped up by a factor of \sqrt{m} . This speedup should be considered when doing a security analysis of elliptic curves whose coefficients lie in a small subfield.

11. *Curves defined over \mathbb{F}_{2^m} , m Composite*

Galbraith and Smart [GS], expanding on earlier work by Frey [Frey], discuss how the Weil descent might be used to solve the ECDLP for elliptic curves defined over \mathbb{F}_{2^m} where m is composite. More recently, Gaudry, Hess and Smart [GHS] refined these ideas to provide strong evidence that when m has a small divisor l (say, $l=4$), the ECDLP for elliptic curves defined over \mathbb{F}_{2^m} can be solved faster than with Pollard's rho algorithm.

It should be noted that some ECC standards, including the draft ANSI X9.63, explicitly exclude the use of elliptic curves over composite fields. The ANSI X9F1 committee also agreed in January 1999 to exclude the use of such curves in a forthcoming revision of ANSI X9.62.

2.3 Is there a subexponential-time algorithm for ECDLP?

Whether or not there exists a subexponential-time algorithm for the ECDLP is an important unsettled question, and one of great relevance to the security of ECC. It is extremely unlikely that anyone will ever be able to *prove* that no subexponential-time algorithm exists for the ECDLP. (Analogously, it is extremely unlikely that anyone will ever be able to *prove* that no polynomial-time (efficient) algorithm exists for the integer factorization and discrete logarithm problems.) However, much work has been done on the DLP over the past 24 years, and more specifically on the ECDLP over the past 16 years. No subexponential-time algorithm has been discovered for the ECDLP, confirming the widely-held belief that no such algorithm exists.

In particular, Miller [Miller] and J. Silverman and Suzuki [SS2] have given convincing arguments for why the most natural way in which the index-calculus algorithms can be applied to the ECDLP is most likely to fail.

Another very interesting line of attack on the ECDLP, called the *xedni-calculus attack* was recently proposed by J. Silverman [Silverman]. However, it was subsequently shown by a team of researchers including J. Silverman (see Jacobson et al. [JKSST]) that the attack is virtually certain to fail in practice.

A summary of the work done on the ECDLP and further references can be found in the Certicom whitepaper [Certicom].

3 The Challenge Explained

This section gives an overview of some of the mathematics that is relevant to this challenge. The format for the challenge parameters presented in Section 4 is also explained.

For further background on finite fields, consult the books by McEliece [McEliece] and Lidl and Niederreiter [LN]. For further background on elliptic curves, consult the books by Koblitz [Koblitz3] and Menezes [Menezes].

3.1 Elliptic curves over F_2^m - format and examples

3.1.1 The finite field F_2^m

There are many ways to represent the elements of a finite field with 2^m elements. The particular method used in this challenge is called a *polynomial basis representation*.

Let $f(x) = x^m + f_{m-1}x^{m-1} + \dots + f_2x^2 + f_1x + f_0$ (where $f_i \in \{0, 1\}$ for $i = 0, 1, \dots, m-1$) be an irreducible polynomial of degree m over F_2 . That is, $f(x)$ cannot be factored as a product of two polynomials over F_2 , each of degree less than m . The polynomial $f(x)$ is called the *reduction polynomial*.

The finite field F_2^m is comprised of all polynomials over F_2 of degree less than m :

$$F_2^m = \{a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_1x + a_0 : a_i \in \{0, 1\}\}.$$

The field element $a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_1x + a_0$ is usually denoted by the binary string $(a_{m-1} a_{m-2} \dots a_1 a_0)$ of length m , so that

$$F_2^m = \{(a_{m-1} a_{m-2} \dots a_1 a_0) : a_i \in \{0, 1\}\}.$$

Thus the elements of F_2^m can be represented by the set of all binary strings of length m . The multiplicative identity element (1) is represented by the bit string $(00 \dots 01)$, while the zero element (additive identity) is represented by the bit string of all 0's.

The following arithmetic operations are defined on the elements of F_2^m :

- *Addition*: If $a = (a_{m-1} a_{m-2} \dots a_1 a_0)$ and $b = (b_{m-1} b_{m-2} \dots b_1 b_0)$ are elements of F_2^m , then $a + b = c = (c_{m-1} c_{m-2} \dots c_1 c_0)$, where $c_i = (a_i + b_i) \bmod 2$. That is, field addition is performed bitwise.
- *Multiplication*: If $a = (a_{m-1} a_{m-2} \dots a_1 a_0)$ and $b = (b_{m-1} b_{m-2} \dots b_1 b_0)$ are elements of F_2^m , then $a \cdot b = r = (r_{m-1} r_{m-2} \dots r_1 r_0)$, where the polynomial $r_{m-1}x^{m-1} + r_{m-2}x^{m-2} + \dots + r_1x + r_0$ is the remainder when the polynomial $(a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_1x + a_0) \cdot (b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \dots + b_1x + b_0)$ is divided by $f(x)$ over F_2 .
- *Inversion*: If a is a non-zero element in F_2^m , the *inverse* of a , denoted a^{-1} , is the unique element $c \in F_2^m$ for which $a \cdot c = 1$.

Example (*The finite field F_2^4*)

Let $f(x) = x^4 + x + 1$ be the reduction polynomial. Then the elements of F_2^4 are:

(0000)	(1000)	(0100)	(1100)	(0010)	(1010)	(0110)	(1110)
(0001)	(1001)	(0101)	(1101)	(0011)	(1011)	(0111)	(1111)

Examples of the arithmetic operations in F_2^4 are:

- $(1101) + (1001) = (0100)$.
- $(1101) \cdot (1001) = (1111)$.
- $(1101)^{-1} = (0100)$.

3.1.2 Elliptic curves over F_2^m

A (non-supersingular) *elliptic curve* $E(F_2^m)$ over F_2^m defined by the parameters $a, b \in F_2^m, b \neq 0$, is the set of all solutions $(x, y), x, y \in F_2^m$, to the equation

$$y^2 + xy = x^3 + ax^2 + b,$$

together with an extra point O , the *point at infinity*.

The set of points $E(F_2^m)$ forms a group with the following addition rules:

1. $O + O = O$
2. $(x, y) + O = O + (x, y) = (x, y)$ for all $(x, y) \in E(F_2^m)$.
3. $(x, y) + (x, x + y) = O$ for all $(x, y) \in E(F_2^m)$ (i.e., the negative of the point (x, y) is $-(x, y) = (x, x + y)$).
4. (Rule for adding two distinct points that are not inverses of each other)

Let $P = (x_1, y_1) \in E(F_2^m)$ and $Q = (x_2, y_2) \in E(F_2^m)$ be two points such that $x_1 \neq x_2$. Then $P + Q = (x_3, y_3)$, where

$$x_3 = \lambda^2 + \lambda + x_1 + x_2 + a,$$

$$y_3 = \lambda(x_1 + x_3) + x_3 + y_1, \text{ and}$$

$$\lambda = \frac{y_2 + y_1}{x_2 + x_1}.$$

5. (Rule for doubling a point)

Let $P = (x_1, y_1) \in E(F_2^m)$ be a point with $x_1 \neq 0$. (If $x_1 = 0$ then $P = -P$, and so $2P = O$.) Then $2P = (x_3, y_3)$, where

$$x_3 = \lambda^2 + \lambda + a,$$

$$y_3 = x_1^2 + (\lambda + 1)x_3, \text{ and}$$

$$\lambda = x_1 + \frac{y_1}{x_1}.$$

Example (*An elliptic curve over F_2^4*)

Consider the finite field F_2^4 defined by the reduction polynomial $f(x) = x^4 + x + 1$.

$y^2 + xy = x^3 + (0011)x^2 + (0001)$ is an equation for an elliptic curve E over F_2^4 . Here $a = (0011)$ and $b = (0001)$. The solutions over F_2^4 to this equation are:

(0000,0001)	(0001,1100)	(0001,1101)	(1000,0101)	(1000,1101)
(0110,1000)	(0110,1110)	(1100,0101)	(1100,1001)	(1010,0111)
(1010,1101)	(0111,0010)	(0111,0101)	(1111,0000)	(1111,1111)

$E(F_2^4)$ has 16 points, including the point at infinity O . The following are examples of the addition law:

- $(1100, 0101) + (1000, 1101) = (0001, 1101)$.
- $2(1100, 0101) = (0111, 0101)$.

3.1.3 Format for challenge parameters (the F_2^m case)

This subsection describes the conventions used for representing the challenge parameters for elliptic curves over F_2^m . Two types of elliptic curves over F_2^m are included in the challenge: *random curves* and *Koblitz curves*.

Koblitz curves over F_2^m are special types of elliptic curves E defined over F_2 which have exactly 2 points in $E(F_2)$. They were first proposed for use in elliptic curve cryptography by Koblitz [Koblitz2]; see also [Solinas].

Apart from the \sqrt{m} -fold speedup that can be obtained with the improved parallelized Pollard's rho method, there have not been any mathematical discoveries to date to suggest that the ECDLP for randomly generated elliptic curves is any easier or harder than the ECDLP for Koblitz curves.

Challenge parameters (random curves)

- m — the order of the finite field is 2^m .
- $f(x)$ — the reduction polynomial which defines the polynomial basis representation of F_2^m .
- seedE — the seed that was used to generate the parameters a and b (see Algorithm 1 in Section 3.1.4).
- a, b — the field elements which define the elliptic curve $E: y^2 + xy = x^3 + ax^2 + b$.
- seedP — the seed that was used to generate the point P (see Algorithm 3 in Section 3.1.4).
- x_P, y_P — the x - and y -coordinates of the base point P .
- n — the order of the point P ; n is a prime number.

- h — the co-factor h (the number of points in $E(\mathbb{F}_{2^m})$ divided by n).
- seedQ — the seed that was used to generate the point Q (see Algorithm 3 in Section 3.1.4).
- x_Q, y_Q — the x - and y -coordinates of the public key point Q .

Challenge parameters (Koblitz curves)

- m — the order of the finite field is 2^m .
- $f(x)$ — the reduction polynomial which defines the polynomial basis representation of \mathbb{F}_{2^m} .
- a, b — the field elements which define the elliptic curve $E: y^2 + xy = x^3 + ax^2 + b$.
- seedP — the seed that was used to generate the point P (see Algorithm 3 in Section 3.1.4).
- x_P, y_P — the x - and y -coordinates of the base point P .
- n — the order of the point P ; n is a prime number.
- h — the co-factor h (the number of points in $E(\mathbb{F}_{2^m})$ divided by n).
- seedQ — the seed that was used to generate the point Q (see Algorithm 3 in Section 3.1.4).
- x_Q, y_Q — the x - and y -coordinates of the public key point Q .

Data formats

- *Integers* are represented in hexadecimal, the rightmost bit being the least significant bit. Example: The decimal integer 123456789 is represented in hexadecimal as 075BCD15.
- *Field elements* (of \mathbb{F}_{2^m}) are represented in hexadecimal, padded with 0's on the left. Example: Suppose $m = 23$. The field element $a = x^{22} + x^{21} + x^{19} + x^{17} + x^5 + 1$ is represented in binary as (1101010000000000100001), or in hexadecimal as 006A0021.
- *Seeds* for generating random elliptic curves and random elliptic curve points (see Section 3.1.4) are 160-bit strings and are represented in hexadecimal.

3.1.4 Random elliptic curves and points (the \mathbb{F}_{2^m} case)

This subsection describes the method that is used for *verifiably* selecting elliptic curves and points at random. The defining parameters of the elliptic curve or point are defined to be outputs of the one-way hash function SHA-1 (as specified in FIPS 180-1 [SHA-1]). The input seed to SHA-1 then serves as proof (under the assumption that SHA-1 cannot be inverted) that the elliptic curve or point were indeed generated at random.

The following notation is used: $s = \lfloor (m - 1)/160 \rfloor$ and $h = m - 160 \cdot s$.

Algorithm 1: Generating a random elliptic curve over \mathbb{F}_{2^m}

Input: A field size $q = 2^m$.

Output: A 160-bit bit string seedE and field elements $a, b \in \mathbb{F}_{2^m}$ which define an elliptic curve E over \mathbb{F}_{2^m} .

1. Choose an arbitrary bit string seedE of length 160 bits.
2. Compute $H = \text{SHA-1}(\text{seedE})$, and let b_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
3. For i from 1 to s do:
Compute $b_i = \text{SHA-1}((\text{seedE} + i) \bmod 2^{160})$.
4. Let b be the field element obtained by the concatenation of b_0, b_1, \dots, b_s as follows:
$$b = b_0 \parallel b_1 \parallel \dots \parallel b_s.$$
5. If $b = 0$ then go to step 1.
6. Let a be an arbitrary element of \mathbb{F}_{2^m} .
(Note: For a fixed b , there are only 2 essentially different choices for a — other values of a give rise to *isomorphic* elliptic curves. Hence the choice of a is essentially without loss of generality.)
7. The elliptic curve chosen over \mathbb{F}_{2^m} is
$$E : y^2 + xy = x^3 + ax^2 + b.$$
8. Output(seedE, a, b).

Algorithm 2: Verifying that an elliptic curve was randomly generated

Input: A field size $q = 2^m$, a bit string seedE of length 160 bits, and field elements $a, b \in \mathbb{F}_{2^m}$ which define an elliptic curve $E : y^2 + xy = x^3 + ax^2 + b$ over \mathbb{F}_{2^m} .

Output: Acceptance or rejection that E was randomly generated using Algorithm 1.

1. Compute $H = \text{SHA-1}(\text{seedE})$, and let b_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
2. For i from 1 to s do:
Compute $b_i = \text{SHA-1}((\text{seedE} + i) \bmod 2^{160})$.
3. Let b' be the field element obtained by the concatenation of b_0, b_1, \dots, b_s as follows:
$$b' = b_0 \parallel b_1 \parallel \dots \parallel b_s.$$
4. If $b = b'$ then accept; otherwise reject.

Algorithm 3: Generating a random elliptic curve point

Input: Field elements $a, b \in \mathbb{F}_{2^m}$ which define an elliptic curve $E : y^2 + xy = x^3 + ax^2 + b$ over \mathbb{F}_{2^m} . The order of $E(\mathbb{F}_{2^m})$ is $n \cdot h$, where n is a prime.

Output: A bit string seedP, a field element y_U , and a point $P \in E(\mathbb{F}_{2^m})$ of order n .

1. Choose an arbitrary bit string seedE of length 160 bits.

2. Compute $H = \text{SHA-1}(\text{seedP})$, and let x_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
3. For i from 1 to s do:
Compute $x_i = \text{SHA-1}((\text{seedP} + i) \bmod 2^{160})$.
4. Let x_U be the field element obtained by the concatenation of x_0, x_1, \dots, x_s as follows:
$$x_U = x_0 \parallel x_1 \parallel \dots \parallel x_s.$$
5. If the equation $y^2 + x_U y = x_U^3 + ax_U^2 + b$ does not have a solution $y \in \mathbb{F}_{2^m}$, then go to step 1.
6. Select an arbitrary solution $y_U \in \mathbb{F}_{2^m}$ to the equation $y^2 + x_U y = x_U^3 + ax_U^2 + b$.
(Note: this equation will have either 1 or 2 distinct solutions. Hence the choice of y_U is essentially without loss of generality.)
7. Let U be the point (x_U, y_U) .
8. Compute $P = hU$.
9. If $P = \mathcal{O}$ then go to step 1.
10. Output(seedP, y_U, P).

Algorithm 4: Verifying that an elliptic curve point was randomly generated

Input: A field size $q = 2^m$, field elements $a, b \in \mathbb{F}_{2^m}$ which define an elliptic curve $E : y^2 + xy = x^3 + ax^2 + b$ over \mathbb{F}_{2^m} , a bit string seedP of length 160 bits, a field element $y_U \in \mathbb{F}_{2^m}$, and an elliptic curve point $P = (x_P, y_P)$. The order of $E(\mathbb{F}_{2^m})$ is $n \cdot h$, where n is a prime.

Output: Acceptance or rejection that P was randomly generated using Algorithm 3.

1. Compute $H = \text{SHA-1}(\text{seedP})$, and let x_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
2. For i from 1 to s do:
Compute $x_i = \text{SHA-1}((\text{seedP} + i) \bmod 2^{160})$.
3. Let x_U be the field element obtained by the concatenation of x_0, x_1, \dots, x_s as follows:
$$x_U = x_0 \parallel x_1 \parallel \dots \parallel x_s.$$
4. Let U be the point (x_U, y_U) .
5. Verify that U satisfies the equation $y^2 + xy = x^3 + ax^2 + b$.
6. Compute $P' = hU$.
7. If $P \neq P'$ then reject.
8. Accept.

3.2 Elliptic curves over F_p - format and examples

3.2.1 The finite field F_p

Let p be a prime number. The finite field F_p is comprised of the set of integers

$$\{0, 1, 2, \dots, p-1\}$$

with the following arithmetic operations:

- *Addition:* If $a, b \in F_p$, then $a + b = r$, where r is the remainder when $a + b$ is divided by p and $0 \leq r \leq p - 1$. This is known as addition modulo p .
- *Multiplication:* If $a, b \in F_p$, then $a \cdot b = s$, where s is the remainder when $a \cdot b$ is divided by p and $0 \leq s \leq p - 1$. This is known as multiplication modulo p .
- *Inversion:* If a is a non-zero element in F_p , the *inverse* of a modulo p , denoted a^{-1} , is the unique integer $c \in F_p$ for which $a \cdot c = 1$.

Example (*The finite field F_{23}*)

The elements of F_{23} are $\{0, 1, 2, \dots, 22\}$. Examples of the arithmetic operations in F_{23} are:

- $12 + 20 = 9$.
- $8 \cdot 9 = 3$.
- $8^{-1} = 3$.

3.2.2 Elliptic curves over F_p

Let $p > 3$ be a prime number. Let $a, b \in F_p$ be such that $4a^3 + 27b^2 \neq 0$ in F_p . An *elliptic curve* $E(F_p)$ over F_p defined by the parameters a and b is the set of all solutions (x, y) , $x, y \in F_p$, to the equation

$$y^2 = x^3 + ax + b,$$

together with an extra point O , the *point at infinity*.

The set of points $E(F_p)$ forms a group with the following addition rules:

1. $O + O = O$
2. $(x, y) + O = O + (x, y) = (x, y)$ for all $(x, y) \in E(F_p)$.
3. $(x, y) + (x, -y) = O$ for all $(x, y) \in E(F_p)$ (i.e., the negative of the point (x, y) is $-(x, y) = (x, -y)$).
4. (Rule for adding two distinct points that are not inverses of each other)

Let $P = (x_1, y_1) \in E(F_p)$ and $Q = (x_2, y_2) \in E(F_p)$ be two points such that $x_1 \neq x_2$. Then $P + Q = (x_3, y_3)$, where

$$x_3 = \lambda^2 - x_1 - x_2,$$

$$y_3 = \lambda(x_1 - x_3) - y_1, \text{ and}$$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}.$$

5. (Rule for doubling a point)

Let $P = (x_1, y_1) \in E(\mathbb{F}_p)$ be a point with $y_1 \neq 0$. (If $y_1 = 0$ then $P = -P$, and so $2P = \mathcal{O}$.) Then $2P = (x_3, y_3)$, where

$$x_3 = \lambda^2 - 2x_1,$$

$$y_3 = \lambda(x_1 - x_3) - y_1, \text{ and}$$

$$\lambda = \frac{3x_1^2 + a}{2y_1}.$$

Example (An elliptic curve over \mathbb{F}_{23})

$y^2 = x^3 + x + 1$ is an equation for an elliptic curve E over \mathbb{F}_{23} . Here $a = 1$ and $b = 1$. The solutions over \mathbb{F}_{23} to this equation are:

(0, 1)	(0, 22)	(1, 7)	(1, 16)	(3, 10)	(3, 13)	(4, 0)	(5, 4)	(5, 19)
(6, 4)	(6, 19)	(7, 11)	(7, 12)	(9, 7)	(9, 16)	(11, 3)	(11, 20)	(12, 4)
(12, 19)	(13, 7)	(13, 16)	(17, 3)	(17, 20)	(18, 3)	(18, 20)	(19, 5)	(19, 18)

$E(\mathbb{F}_{23})$ has 28 points, including the point at infinity \mathcal{O} . The following are examples of the addition law:

- $(3, 10) + (9, 7) = (17, 20)$.
- $2(3, 10) = (7, 12)$.

3.2.3 Format for challenge parameters (the \mathbb{F}_p case)

This subsection describes the conventions used for representing the challenge parameters for elliptic curves over \mathbb{F}_p .

Challenge parameters

- p — the order of the finite field; p is a prime number.
- seedE — the seed that was used to generate the parameters a and b (see Algorithm 5 in Section 3.2.4).
- a, b — the field elements which define the elliptic curve $E: y^2 = x^3 + ax + b$.
- seedP — the seed that was used to generate the point P (see Algorithm 7 in Section 3.2.4).
- x_P, y_P — the x - and y -coordinates of the base point P .
- n — the order of the point P ; n is a prime number.
- h — the co-factor h (the number of points in $E(\mathbb{F}_p)$ divided by n).
- seedQ — the seed that was used to generate the point Q (see Algorithm 7 in Section 3.2.4).

- x_Q, y_Q — the x - and y -coordinates of the public key point Q .

Data formats

- *Integers* are represented in hexadecimal, the rightmost bit being the least significant bit. Example: The decimal integer 123456789 is represented in hexadecimal as 075BCD15.
- *Field elements* (of F_p) are represented as hexadecimal integers.
- *Seeds* for generating random elliptic curves and random elliptic curve points (see Section 3.2.4) are 160-bit strings and are represented in hexadecimal.

3.2.4 Random elliptic curves and points (the F_p case)

This subsection describes the method that is used for *verifiably* selecting elliptic curves and points at random. The defining parameters of the elliptic curve or point are defined to be outputs of the one-way hash function SHA-1 (as specified in FIPS 180-1 [SHA-1]). The input seed to SHA-1 then serves as proof (under the assumption that SHA-1 cannot be inverted) that the elliptic curve or point were indeed generated at random.

The following notation is used: $t = \lceil \log_2 p \rceil$, $s = \lfloor (t-1)/160 \rfloor$ and $h = t - 160 \cdot s$.

Algorithm 5: Generating a random elliptic curve over F_p

Input: A field size p , where p is a prime.

Output: A 160-bit bit string seedE and field elements $a, b \in F_p$ which define an elliptic curve E over F_p .

1. Choose an arbitrary bit string seedE of length 160 bits.
2. Compute $H = \text{SHA-1}(\text{seedE})$, and let c_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
3. Let W_0 denote the bit string of length h bits obtained by setting the leftmost bit of c_0 to 0. (This ensures that $r < p$.)
4. For i from 1 to s do:
Compute $W_i = \text{SHA-1}((\text{seedE} + i) \bmod 2^{160})$.
5. Let W be the bit string obtained by the concatenation of W_0, W_1, \dots, W_s as follows:
$$W = W_0 \parallel W_1 \parallel \dots \parallel W_s.$$
6. Let w_1, w_2, \dots, w_t be the bits of W from leftmost to rightmost. Let r be the integer
$$r = \sum_{i=1}^t w_i 2^{t-i}.$$
7. Choose arbitrary integers $a, b \in F_p$ such that $r \cdot b^2 \equiv a^3 \pmod{p}$.
(Note: For a fixed $r \neq 0$, there are only 2 essentially different choices for a and b — other values of a and b give rise to *isomorphic* elliptic curves. Hence the choice of a and b is essentially without loss of generality.)
8. If $4a^3 + 27b^2 \equiv 0 \pmod{p}$ then go to step 1.

9. The elliptic curve chosen over \mathbb{F}_p is

$$E : y^2 = x^3 + ax + b.$$
10. Output(seedE, a , b).

Algorithm 6: Verifying that an elliptic curve was randomly generated

Input: A field size p (a prime), a bit string seedE of length 160 bits, and field elements $a, b \in \mathbb{F}_p$ which define an elliptic curve $E : y^2 = x^3 + ax + b$ over \mathbb{F}_p .

Output: Acceptance or rejection that E was randomly generated using Algorithm 5.

1. Compute $H = \text{SHA-1}(\text{seedE})$, and let c_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
2. Let W_0 denote the bit string of length h bits obtained by setting the leftmost bit of c_0 to 0.
3. For i from 1 to s do:
 Compute $W_i = \text{SHA-1}((\text{seedE} + i) \bmod 2^{160})$.
4. Let W' be the bit string obtained by the concatenation of W_0, W_1, \dots, W_s as follows:

$$W' = W_0 \parallel W_1 \parallel \dots \parallel W_s.$$
5. Let w_1, w_2, \dots, w_t be the bits of W' from leftmost to rightmost. Let r' be the integer

$$r' = \sum_{i=1}^t w_i 2^{t-i}.$$
6. If $r' \cdot b^2 \equiv a^3 \pmod{p}$ then accept; otherwise reject.

Algorithm 7: Generating a random elliptic curve point

Input: Field elements $a, b \in \mathbb{F}_p$ which define an elliptic curve $E : y^2 = x^3 + ax + b$ over \mathbb{F}_p . The order of $E(\mathbb{F}_p)$ is $n \cdot h$, where n is a prime.

Output: A bit string seedP, a field element y_U , and a point $P \in E(\mathbb{F}_p)$ of order n .

1. Choose an arbitrary bit string seedP of length 160 bits.
2. Compute $H = \text{SHA-1}(\text{seedP})$, and let c_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
3. Let x_0 denote the bit string of length h bits obtained by setting the leftmost bit of c_0 to 0.
4. For i from 1 to s do:
 Compute $x_i = \text{SHA-1}((\text{seedE} + i) \bmod 2^{160})$.
5. Let x_U be the bit string obtained by the concatenation of x_0, x_1, \dots, x_s as follows:

$$x_U = x_0 \parallel x_1 \parallel \dots \parallel x_s.$$
6. If the equation $y^2 = x_U^3 + ax_U + b$ does not have a solution $y \in \mathbb{F}_p$, then go to step 1.
7. Select an arbitrary solution $y_U \in \mathbb{F}_p$ to the equation $y^2 = x_U^3 + ax_U + b$.
 (Note: this equation will have either 1 or 2 distinct solutions. Hence the choice of y_U is essentially without loss of generality.)

8. Let U be the point (x_U, y_U) .
9. Compute $P = hU$.
10. If $P = \mathbf{O}$ then go to step 1.
11. Output(seedP, y_U , P).

Algorithm 8: Verifying that an elliptic curve point was randomly generated

Input: A field size p (a prime), field elements $a, b \in \mathbb{F}_p$ which define an elliptic curve $E : y^2 = x^3 + ax + b$ over \mathbb{F}_p , a bit string seedP of length 160 bits, a field element $y_U \in \mathbb{F}_p$, and an elliptic curve point $P = (x_P, y_P)$. The order of $E(\mathbb{F}_p)$ is $n \cdot h$, where n is a prime.

Output: Acceptance or rejection that P was randomly generated using Algorithm 7.

1. Compute $H = \text{SHA-1}(\text{seedP})$, and let c_0 denote the bit string of length h bits obtained by taking the h rightmost bits of H .
2. Let x_0 denote the bit string of length h bits obtained by setting the leftmost bit of c_0 to 0.
3. For i from 1 to s do:
 Compute $x_i = \text{SHA-1}((\text{seedE} + i) \bmod 2^{160})$.
4. Let x_U be the bit string obtained by the concatenation of x_0, x_1, \dots, x_s as follows:

$$x_U = x_0 \parallel x_1 \parallel \dots \parallel x_s.$$
5. Let U be the point (x_U, y_U) .
6. Verify that U satisfies the equation $y^2 = x^3 + ax + b$.
7. Compute $P' = hU$.
8. If $P \neq P'$ then reject.
9. Accept.

3.3 Further details about the challenge

This subsection presents some more information about the challenge. Each problem posed is to compute the private key given the elliptic curve parameters, the base point P of order n , and the public key point Q . The *private key* is the *unique* integer l , $0 \leq l \leq n - 1$, such that $Q = lP$. Each problem is therefore an instance of the elliptic curve discrete logarithm problem (ECDLP); see Section 2.

With the exception of the Koblitz curves, all elliptic curves have been chosen randomly in a *verifiable* manner (see Sections 3.1.4 and 3.2.4) — anyone can verify that the elliptic curve parameters were indeed generated at random.

Another interesting feature of the challenge is that the points P and Q having order n were also chosen randomly in a *verifiable* manner (see Sections 3.1.4 and 3.2.4). This means that each particular private key l is presently unknown even to the creators of the challenge!! However, any alleged solution l that is found to a challenge can easily be verified by checking that $Q = lP$. The challenges presented

here therefore adhere to the philosophy expressed by Matt Blaze [Blaze] at Crypto '97 that the solutions to a challenge should be unknown to the creators at the outset of the challenge.

The problems have been separated into two categories:

- (i) elliptic curves over F_{2^m} , and
- (ii) elliptic curves over F_p .

There have not been any mathematical discoveries to date to suggest that the ECDLP for elliptic curves over F_{2^m} is any easier or harder than the ECDLP for elliptic curves over F_p .

For each of these categories, the problems have been further divided into three sub-categories:

- (i) Exercises,
- (ii) Level I Challenges, and
- (iii) Level II Challenges.

These are distinguished by the size of the parameter n , the prime order of the base point P . As the size of n increases, the problem is expected to become harder. By a *k-bit challenge*, we shall mean a challenge whose parameter n has bitlength k .

3.4 Time estimates for exercises and challenges

This subsection provides a *very rough* estimate for the time to solve a k -bit challenge with parameter n . These estimates are for software implementations; we do not assume that any special hardware for parallelized Pollard rho attacks is used.

Recall from Section 2.2 that the distributed version of Pollard's rho algorithm using M processors takes approximately $\sqrt{\pi n} / 2 / M$ steps. Here, each "step" is an elliptic curve addition or double together with some rho-method specific operations such as evaluations of hash functions and/or a membership test. Also recall from Section 2.2 that for binary anomalous curves (Koblitz curves) over F_{2^m} , the number of iterations can be reduced up to a factor of $\sqrt{2m}$, and for all other curves up to a factor of $\sqrt{2}$.

Thus, if a computer can perform l operations per second, then the number of computer days required before a discrete logarithm is found is expected to be roughly

$$\frac{1}{l \times 60 \times 60 \times 24} \times \frac{\sqrt{\pi n}}{2\sqrt{mM}} \approx 10^{-5} \times \frac{\sqrt{n}}{l\sqrt{mM}} \text{ machine days}$$

in the case of binary anomalous curves over F_{2^m} , and

$$\frac{1}{l \times 60 \times 60 \times 24} \times \frac{\sqrt{\pi n}}{2M} \approx 10^{-5} \times \frac{\sqrt{n}}{lM} \text{ machine days}$$

for all other curves.

To illustrate this, consider solving an instance of the ECDLP over $F_{2^{89}}$ with $n \approx 2^{89}$. A fast implementation of elliptic curve operations on a widely available computer, say a Pentium 100, may perform 16000 iterations per second for a curve over $F_{2^{89}}$. Thus such an implementation would require

$$10^{-5} \times \frac{\sqrt{2^{89}}}{16000 \times M} \approx \frac{15550}{M} \text{ machine days}$$

to find a single discrete logarithm. So, for example, one such machine running 24 hours a day would require 15550 days. A network of 3000 such machines would require about 5 days.

When estimating the challenge problems, we also take into account that the iterations scale quadratically on the number of machine words required by the field. We assume that we work with a 32-bit machine. Then, for example, a 109-bit field requires 4 machine words while a 89-bit field requires only 3 machine words. This means that each iteration in a 109-bit field should cost $(4/3)^2$ as much. Hence, a Pentium 100 can perform about $l = (3/4)^2 \times 16000 = 9000$ iterations per second for a curve over $F_{2^{109}}$.

For a curve over a 89-bit *prime* field, we estimate that a Pentium 100 can perform about $l = 48000$ iterations per second. For a Koblitz curve over $F_{2^{89}}$, we estimate that a Pentium 100 can perform about $l=24000$ iterations per second. Here we assume that iterations are done on orbits of points rather than on points. This slightly increases the time needed for one iteration while it considerably reduces the expected number of iterations.

The 109-bit Level I challenges are feasible using a very large network of computers. The 131-bit Level I challenges are expected to be infeasible against realistic software and hardware attacks, unless of course, a new algorithm for the ECDLP is discovered.

The Level II challenges are infeasible given today's computer technology and knowledge. The elliptic curves for these challenges meet the stringent security requirements imposed by existing and forthcoming ANSI banking standards [X962, X963].

An implementation report of the Pollard rho algorithm for solving the ECDLP can be found in [HMV]. An implementation report of the solution of some of the exercises can be found in [Escott].

The estimates are included in the tables in section 6.2.

4 Exercise Lists and Challenge Lists

4.1 Elliptic curves over F_{2^m}

In the following tables, ECC2- k denotes that the exercise or challenge is over a field F_{2^m} , and that the parameter n has bitlength k . Furthermore, ECC2K- k denotes that the elliptic curve used is a Koblitz curve (see Section 3.1.3), rather than a randomly generated curve.

For a description of the format of the challenge parameters, see Section 3.1.3. For further details about the challenge, see Section 3.3. The time estimates for each exercise and challenge were derived as in Section 3.4.

Using these timings, it is expected that the 79-bit exercise could be solved in a matter of hours, the 89-bit exercise could be solved in a matter of days, and the 97-bit exercise in a matter of weeks using a network of 3000 computers.

The 109-bit Level I challenge is feasible using a very large network of computers. The 131-bit Level I challenge is expected to be infeasible against realistic software and hardware attacks, unless of course a new algorithm for the ECDLP is discovered.

The Level II challenges are infeasible given today's computer technology and knowledge. The elliptic curves for these challenges meet the stringent security requirements imposed by existing and forthcoming ANSI banking standards [X962, X963].

4.1.1 Exercises

Exercise	Field size (in bits)	Estimated number of machine days	Prize (US\$)
ECC2-79	79	352	<i>Handbook of Applied Cryptography & Maple V software</i>
ECC2-89	89	11278	<i>Handbook of Applied Cryptography & Maple V software</i>
ECC2K-95	97	8637	\$ 5,000
ECC2-97	97	180448	\$ 5,000

4.1.2 Level I challenges

Challenge	Field size (in bits)	Estimated number of machine days	Prize (US\$)
ECC2K-108	109	1.3×10^6	\$ 10,000
ECC2-109	109	2.1×10^7	\$ 10,000
ECC2K-130	131	2.7×10^9	\$ 20,000
ECC2-131	131	6.6×10^{10}	\$ 20,000

4.1.3 Level II challenges

Challenge	Field size (in bits)	Estimated number of machine days	Prize (US\$)
ECC2-163	163	6.2×10^{15}	\$ 30,000
ECC2K-163	163	3.2×10^{14}	\$ 30,000
ECC2-191	191	1.0×10^{20}	\$ 40,000
ECC2-238	239	2.1×10^{27}	\$ 50,000
ECC2K-238	239	9.2×10^{25}	\$ 50,000

ECC2-353	359	1.3×10^{45}	\$ 100,000
ECC2K-358	359	2.8×10^{44}	\$ 100,000

4.2 Elliptic curves over F_p

In the following tables, ECCp- k denotes that the exercise or challenge is over a field F_p (p prime), and that the parameter n has bitlength k .

For a description of the format of the challenge parameters, see Section 3.2.3. For further details about the challenge, see Section 3.3. The time estimates for each exercise and challenge were derived as in Section 3.4.

Using these timings, it is expected that the 79-bit exercise could be solved in a matter of hours, the 89-bit exercise could be solved in a matter of days, and the 97-bit exercise in a matter of weeks using a network of 3000 computers.

The 109-bit Level I challenge is feasible using a very large network of computers. The 131-bit Level I challenge is expected to be infeasible against realistic software and hardware attacks, unless of course a new algorithm for the ECDLP is discovered.

The Level II challenges are infeasible given today's computer technology and knowledge. The elliptic curves for these challenges meet the stringent security requirements imposed by existing and forthcoming ANSI banking standards [X962, X963].

4.2.1 Exercises

Exercise	Field size (in bits)	Estimated number of machine days	Prize (US\$)
ECCp-79	79	146	<i>Handbook of Applied Cryptography & Maple V software</i>
ECCp-89	89	4360	<i>Handbook of Applied Cryptography & Maple V software</i>
ECCp-97	97	71982	\$ 5,000

4.2.2 Level I challenges

Challenge	Field size (in bits)	Estimated number of machine days	Prize (US\$)
ECCp-109	109	9.0×10^6	\$ 10,000
ECCp-131	131	2.3×10^{10}	\$ 20,000

4.2.3 Level II challenges

Challenge	Field size (in bits)	Estimated number of machine days	Prize (US\$)
ECCp-163	163	2.3×10^{15}	\$ 30,000

ECCp-191	191	4.8×10^{19}	\$ 40,000
ECCp-239	239	1.4×10^{27}	\$ 50,000
ECCp-359	359	3.7×10^{45}	\$ 100,000

5 Challenge Rules

5.1 The Rules and Reporting a Solution

Each exercise and challenge in the Exercise and Challenge Lists is based on the problem of computing the ECC private key from the given ECC public key and associated system parameters. An individual or group of individuals reporting a solution must also provide a full explanation of how that solution was reached. No reported solutions will be accepted without a detailed explanation of the steps taken and calculations made to find an ECC private key.

As noted in Section 3.3, each particular private key is presently unknown even to the creators of the Certicom ECC Challenge. Unique to all algorithms based on the discrete logarithm problem, a supposed ECC public key can be validated to ensure it conforms to the arithmetic requirements of a public-key. This validation is 100%. When an ECC public key is validated, it is known that a private key for the public key can logically exist. This capability of key validation is used in the Certicom ECC Challenge.

The proposed solution must be sent via email to Certicom Corp., following the Format of Submissions specified in Section 5.1.1. The correct solution for an Exercise or Challenge will be the one that was received first by Certicom Corp. and checked by an independent, third-party appointed by Certicom.

Certicom Corp. reserves the right to change the contest rules at any time at its sole discretion, without notice, including the right to change or extend the challenge lists, to change the prize amounts, and/or to terminate the contest. While Certicom has appointed an independent, third-party to check the solutions, Certicom Corp. is the sole arbiter and administrator for this contest. Certicom's judgement in all matters is final.

Queries on the Certicom ECC Challenge can be addressed to:

Certicom ECC Challenge Administrator
 Certicom Corp.
 5520 Explorer Drive
 Mississauga, Ontario
 Canada L4W 5L1

For further information concerning the Certicom ECC Challenge, email inquiries can be sent to certicom-ecc-challenge@certicom.com. For news of the latest developments in the Certicom ECC Challenge, check Certicom's web site at www.certicom.com.

5.1.1 Format of Submissions

All solution submissions for any of the exercises or challenges must be sent by email to certicom-ecc-challenge@certicom.com. The report of a solution should clearly state that the submission is being made

for the Certicom ECC Challenge. The body of the email message must contain the following information, titled with the respective headers:

- **Name:** name(s) of the person or people making the submission;
- **Address:** mailing address of the reporting party;
- **Email:** email address of the reporting party;
- **Phone:** telephone number and area code of the reporting party;
- **Exercise or Challenge:** specific exercise or challenge for which the submission is being made (see Sections 4.1 and 4.2 for exercise and challenge tables);
- **Solution:** actual private key value being submitted;
- **Method:** steps and computations taken to calculate the private key, and any other relevant information such as the estimated time taken to calculate the solution and the type of machine(s) used in the computations.

After each field, there must be the word “DONE” to indicate the end of the submission. The “name”, “address”, “email”, “phone”, “exercise or challenge”, “solution”, and “method” fields must be present in every submission. Without these fields, the solution report will be rejected.

While it is preferred that the information fields be separated as specified above, information from two fields can be merged into one. Each field must start on a new line. If more than one person is reporting a solution in a group, the names of each individual along with their corresponding address, email and phone number should be contained in separate fields in alphabetical order.

5.1.2 Administration and Collection of Prizes

The first person or party to report the correct solution for any exercise or challenge, complete with the methodology and steps used to discover that solution, will win the prize for that particular exercise or challenge he/she has solved.

An organized group of individuals reporting a solution will be treated the same as one person reporting a solution, in that only one cash prize will be awarded to the group with the correct solution, reported as specified in section 5.1.1. The prize shall be administered so that it is divided evenly among all members of that group.

In several instances, there are two exercises or challenges with the same field size (e.g. 97-bit exercise) and the same corresponding cash prize, but are based on one of two finite fields—elliptic curves over the finite field F_2^m and elliptic curves over the finite field F_p . These exercises and challenges have different solutions and the corresponding prizes will be awarded accordingly. Therefore, should the correct solution be properly reported for the Exercise ECC2-97 (97-bit field size over the field F_2^m), the ECC2K-97 exercise (97-bit field size over the field F_p) would still be available to solve and the cash prize available for award to the person(s) with the correct solution.

6 Status of the Certicom ECC Challenge

The challenge was released on November 6, 1997, at 1 p.m. EST. The table below shows which problems have been solved so far. Here, the date given as end date indicates the day of submission of the solution to Certicom.

Challenge	End Date	Elliptic Curve Operations	Iterations per second*	Machine Days*
ECC2-79	Dec. 16, 1997	1.7×10^{12}	170000	116
ECC2-89	Feb. 9, 1998	1.8×10^{13}	187000	1114
ECC2K-95	May 21, 1998	2.2×10^{13}	149000	1709
ECC2-97	Sep. 22, 1999	1.2×10^{14}	227000	6118
ECC2K-108	Apr 4, 2000	2.3×10^{15}	160364	166000
ECCp-79	Dec. 6, 1997	1.4×10^{12}	314000	52
ECCp-89	Jan. 12, 1998	2.4×10^{13}	388000	716
ECCp-97	Mar. 18, 1998	2.0×10^{14}	361000	6412

*on 500 MHz Digital Alpha workstation running Linux

Since the algorithms for all problems were based on Pollard's rho method [Pollard], the number of elliptic curve operations indicated above is the same as the number of iterations in the rho method.

The fourth column of the table indicates how many such iterations per seconds were performed by the challenge solvers' routines on a 500 MHz Alpha workstation, which is a typical machine used in the computations. Here, any speedup going from smaller to larger field sizes is due to code optimization. The penultimate column shows how many machine days on a 500 MHz Alpha would have been necessary if the whole computation had been performed on a single such machine running 24 hours a day.

All problems except the ECCp-97 problem were solved using the parallelized Pollard rho method due to van Oorschot and Wiener [VW]; for the ECCp-97 problem, a parallelized Brent-type cycle-finding algorithm was used. Furthermore, for the ECC2K-95 and ECC2K-108 problem the number of iterations was reduced since the iterations were performed on orbits rather than on individual points.

For an implementation report of the solution of some of the exercises, see [Escott].

6.1 Details of solved problems

6.1.1 ECC2-79

- Solution: $\log_p Q = 3AA0\ 68A09F1E\ D21E2582$
- Date: 16 Dec 1997
- Communicated by: Robert J. Harley (Robert.Harley@inria.fr)
- Who solved it: group of about 20 people
- Method: parallelized Pollard rho method:
 - 1 737 410 165 382 iterations
 - 1 617 distinguished points

6.1.2 ECC2-89

- Solution: $\log_p Q = 22A2A8\ 8267271F\ 48147AA5$

- Date: 9 Feb 1998
- Communicated by: Robert J. Harley (Robert.Harley@inria.fr)
- Who solved it: group of about 70 people
- Method: parallelized Pollard rho method:
 - 18 161 819 582 507 iterations
 - 17 543 distinguished points

6.1.3 ECC2K-95

- Solution: $\log_p Q = 7A4249BA\ 547313AF\ 5D482092$
- Date: 21 May 1998
- Communicated by: Robert J. Harley (Robert.Harley@inria.fr)
- Who solved it: group of 47 people
- Method: parallelized Pollard rho method, with speedup by introduction of equivalence classes:
 - Roughly 21 600 000 000 000 iterations
 - 74 268 distinguished points
 - equivalence classes: the orbits under the involution $[-1]$ and the Frobenius endomorphism $[(-1 + \sqrt{-7})/2]$ (= clusters of 194 points).

6.1.4 ECC2-97

- Solution: $\log_p Q = CF765639\ D6B81943\ FD89D60F$
- Date: 22 Sep 1999
- Communicated by: Robert J. Harley (Robert.Harley@inria.fr)
- Who solved it: group of 195 people
- Method: parallelized Pollard rho method:
 - 119 248 522 782 547 iterations
 - 127 492 distinguished points

6.1.5 ECC2K-108

- Solution: $\log_p Q = 0256\ F98379F7\ 214306BE\ 5BE678FD$
- Date: 4 Apr 2000
- Communicated by: Robert J. Harley (Robert.Harley@inria.fr)
- Who solved it: group of about 1300 people
- Method: parallelized Pollard rho method with speedup:
 - 2.3×10^{15} iterations
 - 2.05 million distinguished points

6.1.6 ECCp-79

- Solution: $\log_p Q = 1387\ 56822DD5\ FB093766$
- Date: 6 Dec 1997
- Communicated by: Robert J. Harley (Robert.Harley@inria.fr)
- Who solved it: W. Baisley and R.J. Harley
- Method: Pollard's rho method, parallelized Brent-type cycle-finding algorithm:
 - About 1 400 000 000 000 iterations

6.1.7 ECCp-89

- Solution: $\log_p Q = 0113C284 D9BD7B58 BCA30C67$
- Date: 12 Jan 1998
- Communicated by: Robert J. Harley (Robert.Harley@inria.fr)
- Who solved it: group of 57 people
- Method: parallelized Pollard rho method:
 - 24 249 418 904 337 iterations
 - 36 345 distinguished points

6.1.8 ECCp-97

- Solution: $\log_p Q = 01 6C86AA7C ACF69F1D D28B3E2F$
- Date: 18 Mar 1998
- Communicated by: Dimitris Tsapakidis (dimitris@alien.bt.co.uk)
- Who solved it: group of 588 people
- Method: parallelized Pollard rho method:
 - About 2.0×10^{14} iterations
 - 186 364 distinguished points

6.2 Status and estimates of ECC Challenge problems

Here we list the tables of both solved and unsolved problems, their status and Certicom's estimate of computation required for solution.

For a detailed explanation of the estimates, please refer to section 3.4.

6.2.1 Elliptic Curves over F_{2^m}

Exercises						
Challenge	Start Date	End Date	Number of Elliptic Curve Operations taken	Estimated number of		
				Elliptic Curve Operations	Iterations per second*	Machine days*
ECC2-79	Nov. 6, 1997	Dec. 16, 1997	1.7×10^{12}	4.9×10^{11}	16000	352
ECC2-89	Nov. 6, 1997	Feb. 9, 1998	1.8×10^{13}	1.6×10^{13}	16000	11278
ECC2K-95	Nov. 6, 1997	May. 21, 1998	2.16×10^{13}	1.8×10^{13}	24000	8637
ECC2-97	Nov. 6, 1997	Sep. 22, 1999	1.2×10^{14}	2.5×10^{14}	16000	180448

Level I Challenges						
Challenge	Start Date	End Date	Number of Elliptic Curve Operations taken	Estimated number of		
				Elliptic Curve Operations	Iterations per second*	Machine days*
ECC2-109	Nov. 6, 1997	Nov. , 2002		1.6×10^{16}	9000	2.1×10^7
ECC2K-108	Nov. 6, 1997	Apr. 4, 2000	2.3×10^{15}	1.5×10^{15}	13500	1.3×10^6
ECC2-131	Nov. 6, 1997			3.3×10^{19}	5760	6.6×10^{10}
ECC2K-130	Nov. 6, 1997			2.0×10^{18}	8640	2.7×10^9

Level II Challenges						
Challenge	Start Date	End Date	Number of Elliptic Curve Operations taken	Estimated number of		
				Elliptic Curve Operations	Iterations per second*	Machine days*
ECC2-163	Nov. 6, 1997			2.1×10^{24}	4000	6.2×10^{15}
ECC2K-163	Nov. 6, 1997			1.7×10^{23}	6000	3.2×10^{14}
ECC2-191	Nov. 6, 1997			3.5×10^{28}	4000	1.0×10^{20}
ECC2-238	Nov. 6, 1997			4.2×10^{35}	2250	2.1×10^{27}
ECC2K-238	Nov. 6, 1997			2.7×10^{34}	3375	9.2×10^{25}
ECC2-353	Nov. 6, 1997			1.1×10^{53}	1000	1.3×10^{45}
ECC2K-358	Nov. 6, 1997			3.6×10^{52}	1500	2.8×10^{44}

*Estimation based on a Pentium 100

6.2.2 Elliptic curves over F_p

Exercises						
Challenge	Start Date	End Date	Number of Elliptic Curve Operations taken	Estimated number of		
				Elliptic Curve Operations	Iterations per second*	Machine days*
ECCp-79	Nov. 6, 1997	Dec. 6, 1997	1.4×10^{12}	6.1×10^{11}	48000	146
ECCp-89	Nov. 6, 1997	Jan. 12, 1998	2.4×10^{13}	1.8×10^{13}	48000	4360
ECCp-97	Nov. 6, 1997	Mar. 18, 1998	2.0×10^{14}	3.0×10^{14}	48000	71982

Level I Challenges						
Challenge	Start Date	End Date	Number of Elliptic Curve Operations taken	Estimated number of		
				Elliptic Curve Operations	Iterations per second*	Machine days*
ECCp-109	Nov. 6, 1997			2.1×10^{16}	27000	9.0×10^6
ECCp-131	Nov. 6, 1997			3.5×10^{19}	17820	2.3×10^{11}

Level II Challenges						
Challenge	Start Date	End Date	Number of Elliptic Curve Operations taken	Estimated number of		
				Elliptic Curve Operations	Iterations per second*	Machine days*
ECCp-163	Nov. 6, 1997			2.4×10^{24}	12000	2.3×10^{15}
ECCp-191	Nov. 6, 1997			4.9×10^{28}	12000	4.8×10^{19}
ECCp-239	Nov. 6, 1997			8.2×10^{35}	6750	1.4×10^{27}
ECCp-359	Nov. 6, 1997			9.6×10^{53}	3000	3.7×10^{45}

*Estimation based on a Pentium 100

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